Experiment 1

Measurement of Density

1. Purpose

To learn how to measure physical quantities with the vernier caliper and the micrometer caliper and how to analyze data with the propagation of uncertainty.

2. Introduction

Measurement is an experimental process of obtaining the value of a physical quantity. Experience has shown that no measurement, however carefully made, can be completely free of uncertainties. For the whole structure and application of science depends on measurements, the ability to evaluate these uncertainties and keep them to a minimum is crucially important. In other words, an experiment is not complete until an analysis of the uncertainty in the final result to be reported has been conducted. Therefore, this experiment focuses on the measurement of densities, where the propagation of uncertainties should be considered since we cannot obtain the density of the measurand directly, with the hope that students can get familiar with the techniques of data analysis.

3. Theory

Uncertainty analysis is the evaluation of uncertainty in measurement. The word *uncertainty* in science does not carry the meanings of the terms *mistake* or *blunder*. In contrast, it would inevitably occur to all measurements. As such, **uncertainties are not mistakes; you cannot eliminate them by being very careful.** The best you can hope to do is to ensure the errors are as small as reasonably possible and to have a reliable estimate of how large they are.

3.1 Standard Form for Stating Uncertainties[1]

In general, since we do not know the answer before measurements, it is only an **estimate** of the value of the measurand and thus is complete only when accompanied by a statement of the uncertainty of that estimate. The standard form for reporting a measurement of a physical quantity x is

$$(\text{measured value of } x) = x_{\text{best}} \pm \delta x, \tag{3.1}$$

[1] John R. Taylor, An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements, 2nd ed. (University Science Books, Sausalito, 1997).

where

$$x_{\text{best}} = \text{(best estimate for } x\text{)}$$

and

$$\delta x = \text{(an estimate of an uncertainty in the measurement)}$$

Also, the difference between two measured values of the same quantity is the *discrepancy*. You may find the meaning of the range $x_{\text{best}} - \delta x$ to $x_{\text{best}} + \delta x$ somewhat vague, thinking it tells us we are absolutely certain the measured quantity lies in a range. Unfortunately, in most scientific measurements, such a statement is hard to make. **We cannot state percent confidence in our margins of uncertainty until we understand the statistical laws that govern the process of measurement.** We will return to this point later. For now, let us be content with defining the uncertainty δx so that we are "reasonably certain" the measured quantity lies between $x_{\text{best}} - \delta x$ and $x_{\text{best}} + \delta x$.

3.1.1 Significant Figures

Because the quantity δx is an estimate of uncertainty, obviously it should not be stated with too much precision. If we measure the acceleration of gravity g, it would be absurd to state a result like

$$\text{(measured } g) = 9.821 \pm 0.02325 \ {}^{m}\!/\!{}_{s^2} \tag{3.2}$$

The uncertainty in the measurement cannot conceivably be known to four significant figures. Instead, uncertainties should be stated with only one or two significant figures for more precise uncertainty has no meaning, noting that we usually choose to state the uncertainties with **two significant figures** in high-precision work. Thus, if some calculation yields the uncertainty $\delta g = 0.02385 \ {}^{m}\!/\!{}_{s^2}$, this answer should **be rounded up to** $\delta g = 0.024 \ {}^{m}\!/\!{}_{s^2}$, and the conclusion (3.2) should be rewritten as

$$\text{(measured } g) = 9.821 \pm 0.024 \ {}^{m}\!/\!{}_{s^2} \tag{3.3}$$

Once the uncertainty in a measurement has been estimated, the significant figures in the measured value must be considered. A statement such as

$$\text{measured speed} = 6051.78 \pm 30 \ {}^{m}\!/\!{}_{s} \tag{3.4}$$

is obviously ridiculous. The uncertainty of 30 means that the digit 5 might really be as small as 2 or as large as 8. Clearly, the trailing digits 1,7, and 8 have no significance at all and should be rounded. That is, the correct statement of (3.4) is

$$\text{measured speed} = 6050 \pm 30 \ {}^{m}\!/\!_{s} \tag{3.5}$$

The general rule for stating answers is that the last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty. Note that the uncertainty in any measured quantity has the same dimensions as the measured quantity itself. Therefore, writing the units ($\mathrm{m/s}$, cm^3 , etc.) after both the answer and the uncertainty is clearer and more economical. By the same token, if a measured number is so large or small that it calls for **scientific notation** (the use of the form 3×10^3 instead of 3000, for example), then it is simpler and clearer to put the answer and uncertainty in the same form. For example,

$$\text{measured charge } = (1.61 \pm 0.05)\times10^{-19} \ \text{ coulombs}$$

is much easier to read and understand than in the form

$$\text{measured charge } = 1.61\times10^{-19} \pm 0.05\times10^{-21} \ \text{ coulombs}$$

3.1.2 Fractional Uncertainty

If x is measured in the standard form $x_{best} \pm \delta x$, the fractional uncertainty in x is

$$\text{fractional uncertainty } = \frac{\delta x}{|x_{\text{best}}|} \tag{3.6}$$

and the *percent uncertainty* is just the fractional uncertainty expressed as a percentage (that is, multiplied by 100%). For example, the result (3.5) can be rewritten as

$$\text{measured speed} = 6050 \, {}^{m}\!/\!_{s} \pm 0.0050 \tag{3.7}$$

or

$$\text{measured speed} = 6050 \, {}^{m}\!/\!_{s} \pm 0.50\% \tag{3.8}$$

Note that $\delta x / |x_{\text{best}}|$ is a dimensionless quantity and keep in mind that for quantities that are very hard to measure, a 10% uncertainty would be regarded as an experimental triumph. Large percentage uncertainties do not necessarily mean that measurement is scientifically useless.

Also, as you relate fractional uncertainty with the idea of significant figures, you should understand why no more than two significant figures should be stated for the uncertainties.

3.1.3 Propagation of Uncertainties

Most physical quantities usually cannot be measured in a single direct measurement but are instead found in two distinct steps. For example, to find the momentum p of a car, we should first measure its mass m and its velocity v, and then use these values to calculate its momentum. To do so, we unavoidably have to estimate the uncertainties in the quantities measured directly and then determine how these uncertainties $(\delta m,\ \delta v)$ "propagate" through the calculations to produce an uncertainty in the final answer (δp). Here, we would only give the rules of propagation of uncertainties instead of providing a rigorous proof due to the complexity. However, you are encouraged to come back and study the reasons why by yourself after learning more about the statistics. For now, let's focus on how to deal with the propagation.

Suppose that two **independent and random quantities** x and y are measured with uncertainties $\delta x, \delta y$. We have uncertainty in sum and difference to be

$$\delta(x \pm y) = \sqrt{(\delta x)^2 + (\delta y)^2} \tag{3.9}$$

in product and quotient to be

$$\frac{\delta(x/y)}{\left|\overline{x/y}\right|} = \frac{\delta(xy)}{\left|\overline{xy}\right|} = \sqrt{\left(\frac{\delta x}{\overline{x}}\right)^2 + \left(\frac{\delta y}{\overline{y}}\right)^2} \tag{3.10}$$

and in powers to be

$$\frac{\delta\left(x^y\right)}{\left|\overline{x^y}\right|} = |y| \frac{\delta x}{\left|\overline{x}\right|} \tag{3.11}$$

In general, for n independent and random quantities, the uncertainty is the quadratic sum

$$\delta(x_1 + \cdots + x_n) = \sqrt{(\delta x_1)^2 + \cdots + (\delta x_n)^2} \tag{3.12}$$

$$\frac{\delta\left(\frac{x_1 \times \cdots \times x_n}{y_1 \times \cdots \times y_n}\right)}{\left|\overline{\frac{x_1 \times \cdots \times x_n}{y_1 \times \cdots \times y_n}}\right|} = \sqrt{\left(\frac{\delta x_1}{\overline{x_1}}\right)^2 + \cdots + \left(\frac{\delta x_n}{\overline{x_n}}\right)^2 + \left(\frac{\delta y_1}{\overline{y_1}}\right)^2 + \cdots + \left(\frac{\delta y_n}{\overline{y_n}}\right)^2} \tag{3.13}$$

3.1.4 Classification of Uncertainties

So far, we have discussed how to state and how to propagate uncertainties in a standard way. While facing repeated observations with different results, it is natural to ask ourselves which value is the most representative and what confidence level can we have in that value and the method we use is to introduce the best estimate as well as the uncertainty to state the result. For n independent determinations X_k of X, the best estimate is usually taken as the arithmetic mean or average of n independent determinations, that is

$$X_{\text{best}} = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{3.14}$$

As for the uncertainties, according to the International Standard Organization (ISO)[2], uncertainties that occur can be classified into two types, called type A and type B:

(i) ***Type A standard uncertainties***, evaluated through the statistical analysis considering the random effects which make the individual observations X_k differ in value, are defined to be the standard deviations of the averages. The experimental variance of the observations, estimating the variance σ^2 of the probability distribution of X_k, is given by

$$\sigma_X{}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2 \tag{3.15}$$

This estimate of variance and its positive square root σ_X, termed the **experimental standard deviation**, characterize the variability of the observed values X_k, or more specifically, their dispersion about their mean $\overline{X}$. Now, to avoid confusion, it's better to define another random variable A_X, the averages obtained by different and independent trials. Therefore, the best estimate of the variance of the mean is given by

$$Var\left(A_X\right) = \sigma_{A_X}{}^2 = E\left[A_X{}^2\right] - \left(E\left[A_X\right]\right)^2 = \frac{\sigma_X{}^2}{n} \tag{3.16}$$

where $E\left[X\right]$ stands for the expectation value for the quantity X. The Type A standard uncertainty $u_{\mathrm{A}}\left(X\right)$ is therefore obtained by

$$u_A\left(X\right) = \sqrt{Var\left(A_X\right)} = \sigma_{A_X} = \frac{\sigma_X}{\sqrt{n}} = \sqrt{\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2}{n\left(n-1\right)}} \tag{3.17}$$

Note that in (3.15), the standard deviation should be defined by the factor $N-1$ instead of N due to Bessel correction which will not be proven here, and that (3.16)

[2] BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP, and OIML, Evaluation of Measurement Data—Guide to the Expression of Uncertainty in Measurement (International Organization for Standardization, Geneva, 2008

stands only if X_k are n independent observations of the quantity X. Also, as expected, the best estimate of the variance of the mean $u_{\mathrm{A}}(X)$ approaches to 0, as long as the number of trials n is large enough when the random effect would averagely not influence the measurements at all.

(ii) ***Type B standard uncertainties*** are evaluated by non-statistical information such as instrument characteristics considering the systematic effects which result in discrepancies between the measurand values and the reference value that remain constants or change predictably in replicated measurements. The pool of information may include previous measurement data, manufacturer's specifications, data provided in calibration, uncertainties assigned to reference data taken from handbooks, or simply the experience.

If the estimate X_k is taken from a manufacturer's specification, calibration certificate, handbook, or other source and its quoted uncertainty is stated to be a particular multiple of a standard deviation, the standard uncertainty $u_B(X_k)$ is simply the quoted value divided by the multiplier, and the estimated variance $u_B{}^2(X_k)$ is the square of that quotient. For example, a calibration certificate states that the mass of a stainless steel mass standard m_s of nominal value one kilogram is $1000.000325\,g$ and that "the uncertainty of this value is $240\mu g$ at the three standard deviation level." The standard uncertainty of the mass standard is then simply

$$u_B(m_s) = \frac{240\ \mu g}{3} = 80\ \mu g \tag{3.18}$$

On the other hand, if the uncertainty is not provided by the manufacturer, it can still be roughly calculated by

$$u_B(X) = \frac{a}{2\sqrt{3}} \tag{3.19}$$

where a is the minimum scale value of the instrument. Note that in (3.19), we have assumed that it is equally for X_k to lie anywhere within the interval $\overline{X} - \frac{a}{2}$ to $\overline{X} + \frac{a}{2}$ (a uniform or rectangular distribution of possible values) for practical purposes.

Last but not the least, after obtaining Type A uncertainty and Type B uncertainty, the combined standard uncertainty $u_c(X)$ for independent input quantities X_k is therefore determined by

$$u_C(X) = \sqrt{u_A{}^2(X) + u_B{}^2(X)} = \delta X \tag{3.20}$$

where δX is the best estimate of the uncertainty in the measurement of X_k.

3.1.5 Example: Measurement of the volume of a cubic block

To obtain the volume, the side length of a cubic block should be measured first, and the results L_k are shown in Table1 with the minimum scale value of the ruler to be $1\ mm$.

Table1. measured values of the side length

No	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Value(mm)	22.1	22.0	21.9	21.8	21.8	21.7	21.9	22.0
No	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Value(mm)	21.9	22.0	21.9	22.1	21.9	21.8	22.0	21.8

(i) Best estimate for the side length: (by 3.14)

$$L_{\text{best}} = \overline{L} = \frac{\sum_{i=1}^{16} L_i}{16} = 21.931\ldots \approx 21.931 \text{ (mm)}$$

(ii) Type A standard uncertainty: (by 3.16)

$$u_A(L) = \sqrt{Var(A_L)} = \frac{\sigma_L}{\sqrt{16}} = \sqrt{\frac{\sum_{i=1}^{16}\left(L_i - \overline{L}\right)}{16(16-1)}} = 0.0373... \approx 0.037 \text{ (mm)}$$

(iii) Type B standard uncertainty: (by 3.19)

$$u_B(L) = \frac{1\ (mm)}{2\sqrt{3}} = 0.2886... \approx 0.29 \text{ (mm)}$$

(iv) Combined standard uncertainty: (by 3.20)

$$u_C(L) = \sqrt{u_A{}^2(L) + u_B{}^2(L)} = \delta L \approx 0.291 \approx 0.30 \text{ (mm)}$$

(v) Measured value of the side length:

$$\text{(Measured side length } L) = L_{\text{best}} + \delta L = \overline{L} + u_C(L) = 21.93 \pm 0.30 \text{ (mm)}$$

(vi) Best estimate for the volume of the cubic block:

$$V_{\text{best}} = \overline{L}^3 = (21.93)^3 = 10546.683... \approx 10546.68 \text{ (mm}^3\text{)}$$

(vii) Best estimate for the uncertainty of the volume: (by 3.11)

$$\frac{\delta(V)}{|V_{\text{best}}|} = \frac{\delta\left(\overline{L}^3\right)}{\left|\overline{L}^3\right|} = 3\frac{\delta L}{\left|\overline{L}\right|} = \frac{3 \times 0.30}{21.93} \approx 0.041$$

$$\therefore \delta(V) = 10546.68 \times 0.041 = 432.4... \approx 440 \text{ (mm}^3\text{)}$$

(viii)Calculated volume of the cubic block:

$$V = V_{\text{best}} \pm \delta V = 10546.68 \pm 440 \approx 10547 \pm 440 \ (\text{mm}^3)$$

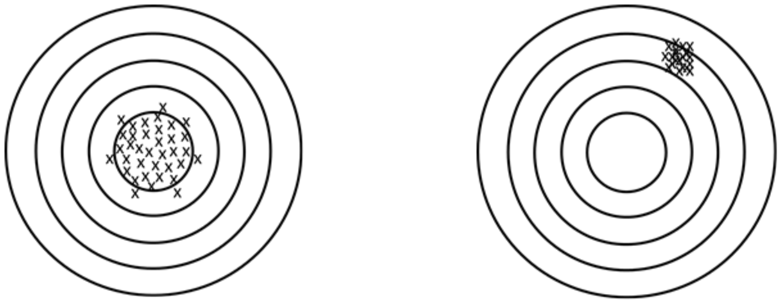
or

$$V = 10547 \ \text{mm}^3 \pm 4.2\%$$

3.2 Statistical Analysis of Random Uncertainties

As noted before, uncertainties are classified into two groups: Type A standard uncertainty or the random uncertainties, which can be treated statistically and be revealed by repeating the measurements, and Type B standard uncertainty or the systematic uncertainties, which cannot. To get a better feel for the difference between random and systematic uncertainties, consider the analogy shown in Figure 1. Here the "experiment" is a series of shots fired at a target; accurate "measurements" are shots that arrive close to the center. Random effect is caused by anything that makes the shots arrive at randomly different points, such as fluctuating atmospheric conditions between the marksman and the target. Systematic effect arises if anything makes the shots arrive off-center in one "systematic" direction, such as misaligned gun slights.

Although Figure 1 is an excellent illustration of the random effect and the systematic effect, it is, however, misleading in one important respect. Because each of the two pictures shows the position of the target, we can tell at a glance whether a particular shot was accurate or not. Nonetheless, in real-life experiments, we do not know the true value of the measurand; that is, we do not know the accurate position of the target, which tells that we can easily assess the random effect but get no guidance concerning the systematic effect in most real experiments.



(a) (b)

Figure 1. Random and systematic effect in target practice. The random effect is larger in (a), compared to (b), and the systematic effect is larger in (b), compared to (a).

Therefore, systematic uncertainties are usually hard to evaluate and even to detect. The experienced scientist has to learn to anticipate the possible sources of systematic effect and to make sure that all systematic effect is much less than the required precision. Also, *the reference value* or *the most probable value* of the best estimate for the measurands relies on differently and independently repeated measurements under the same condition to determine.

3.2.1 Common probability distribution－Normal distribution[3]

[3] Ifan G. Hughes and Thomas P. A. Hase, Measurements and Their Uncertainties: A Practical Guide to Modern Error Analysis (Oxford U. P., Oxford, 2010).

For measurements with random effect, the distribution is called the normal, or Gaussian distribution, also referred to as the "bell curve." Mathematically, it is a two-parameter function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\left(x-\bar{x}\right)^2}{2\sigma^2}\right] \tag{3.21}$$

which describes the distribution of the data about the mean, $\bar{x}$, with standard deviation σ. Many real-life data sets have bell-shaped distribution and are approximately symmetric about the mean for the random effect.

Figure 2 (a) shows about 68%, 95%, 99,7% of the data lie within 1, 2, 3 standard deviation of the mean, or the interval $\bar{x}-\sigma$ to $\bar{x}+\sigma$, $\bar{x}-2\sigma$ to $\bar{x}+2\sigma$, $\bar{x}-3\sigma$ to $\bar{x}+3\sigma$, respectively. Figure2 (b) shows the functional form of three normalized Gaussian distributions, each with standard deviations of $1/2,\ 1,$ and 2, respectively. Each curve has its peak centered on the mean, is symmetric about the value, and has an area under the curves equal to 1. While there are many possible definitions of the "width" of a Gaussian distribution, including Full Width at Half Maximum (FWHM), the $(1/e)$ width, the $\left(1/e^2\right)$ width, each version is proportional to the standard deviation; therefore, we can say that the larger the standard deviation, the broader the distribution, and correspondingly lower the peak value.

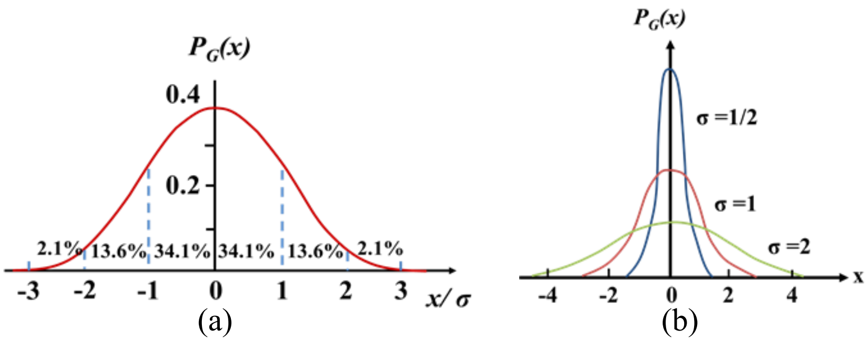


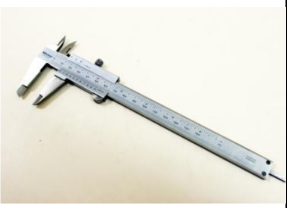
Figure 2. Functional forms of the normalized normal distributions. (a) The percentage of data within the interval $\bar{x}-\sigma$ to $\bar{x}+\sigma$, $\bar{x}-2\sigma$ to $\bar{x}+2\sigma$, $\bar{x}-3\sigma$ to $\bar{x}+3\sigma$, respectively. (b) Gaussian distributions, each with standard deviations of $1/2,\ 1,$ and 2, respectively, and an area under each curve equal to 1.

Recall the claim at the beginning of Section 3.1, that the standard deviation σ_x characterizes the average uncertainty of the measurements X_k. We can now tell that if the same quantity X is measured many times under the same condition, and if all the sources of uncertainty are small and random, then the results will be distributed nearly around the average under the bell-shaped curve. In particular, approximately 68% of your results will fall within a distance σ_X on either side of $\overline{X}$; that is, 68% of your measurements will fall in the range $\overline{X} \pm \sigma_X$. In other words, if you make a single measurement under the same condition, the probability is 68% that your result will be within the interval $\overline{X} - \sigma$ to $\overline{X} + \sigma$. Thus, we can adopt σ_X to mean exactly what we have been calling "uncertainty." If you make one measurement of X, the uncertainty associated with this measurement can be taken to be

$$\delta X = \sigma_X$$

with this choice, you can be 68% confident the measurement is within δX of the best estimate.

4. Apparatus

				
vernier caliper	micrometer caliper	straight ruler	electric balance	precision balance

5. Procedures

(1) Pre-lab assignments (hand in before the experiment with no more than 2 pages A4)

1. Read the instructions for use of the vernier caliper and the micrometer caliper carefully to understand how to use them to measure the quantities
2. Get familiar with the theory and make tables for the experiment in excel
3. Take no more than one page to make a flowchart of this experiment and to summarize the five most important ideas of this material
4. Answer the following questions on the other side of your paper.

   (i) Rewrite each of the following measurements in its most appropriate form

   (a) $v = 8.123456 \pm 0.0312$ m/s

   (b) $x = 3.1234 \times 10^4 \pm 2$ m

   (c) $m = 5.6789 \times 10^{-7} \pm 3 \times 10^{-9}$ kg

   (ii) In an experiment with a simple pendulum, a student decides to check whether the period T is independent of the amplitude A (defined as the largest angle that the pendulum makes with the vertical during its oscillations). He obtains the results shown in the Table .

Amplitude A (deg)	Period T (s)
5 ± 2	1.932 ± 0.005
17 ± 2	1.94 ± 0.01
25 ± 2	1.96 ± 0.01
40 ± 2	2.01 ± 0.01
53 ± 2	2.04 ± 0.01
67 ± 2	2.12 ± 0.02

(a) Draw a graph of T against A. (Consider your choice of scales carefully. If you have any doubt about this choice, draw two graphs, one including the origin, $A=T=0$, and one in which only values of T between 1.9 and 2.2 s are shown.) Should the student conclude that the period is independent of the amplitude?

(b) Discuss how the conclusions of part (a) would be affected if all the measured values of T had been uncertainty by $\pm0.3\text{ s}$.

(2) In-lab activities

1. Calibrate the instruments to avoid the zero-point errors
2. Obtain the densities of objects assigned by the lab instructor.
   (i) Use the electric balance, the straight ruler, the vernier caliper, and the micrometer caliper to independently measure the quantities you need while calculating the densities of the given objects **20 times** and record the data in the excel tables
   (ii) Calculate the average values, the standard deviations, and the average standard deviations of the quantities you measured, and report them in the standard forms
   (iii) Calculate the values of the densities by the propagation of uncertainty, and report the results in the standard forms
   (iv) Use the Archimedean method to obtain the densities of the objects without the straight ruler, the vernier caliper and the micrometer caliper and report the results in the standard forms
   (v) Compare your results of the measurement, and redo the experiments if necessary

(3) Post-lab report

1. Recopy and organize your data from the in-lab tables in a neat and more readable form
2. Analyze the data you obtained in the lab and answer the given questions

6. Questions

(1) While measuring the height and the diameter of a cylinder metal rod, why should you do the procedures at different points of the rod and from different directions each time?

(2) While using the ruler to measure the length of the measurand several times, why should you take different parts of the ruler to experiment each time?

(3) Suppose you are asked to measure a rectangular object. While calculating the value of its area, shall you first obtain the averages of the length and the width by the measured data,

and then calculate the value of the area by the product of them; or shall you instead calculate the values of the area for each measurement first, and then have their average to be the result? Explain.

(4) Discuss the systematic effect that occurs during the experiments. What are they, and how do they influence the results?

(5) Is it possible for you to design a vernier caliper with its accuracy to be 0.02 mm ? Explain.

(6) (3.10) and (3.11) seem to give different results for the uncertainty of x^2. Which one is correct? Explain.

(7) What is the difference between accuracy and precision?

7. Reference

[1]John R. Taylor, An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements, 2nd ed. (University Science Books, Sausalito, 1997).

[2]Ifan G. Hughes and Thomas P. A. Hase, Measurements and Their Uncertainties: A Practical Guide to Modern Error Analysis (Oxford U. P., Oxford, 2010).

[3]BIPM, IEC, IFCC, ILAC, ISO, IUPAC, IUPAP, and OIML, Evaluation of Measurement Data—Guide to the Expression of Uncertainty in Measurement (International Organization for Standardization, Geneva, 2008

[4]StateQuest: Confidence Intervals: https://www.youtube.com/watch?v=TqOeMYtOc1w